MA 681, Spring 2011

Assignment 2

Geometric interpretation of derivative. Elementary functions. Basics of integration

I prefer that you submit this assignment by Wednesday, March 3rd. However, if you are somehow delayed, take your time (just don't get overwhelmed by homeworks piling up.)

Collaboration is welcome. If you do collaborate, make sure to write/type your own paper.

(1) Find an angle by which a curve at z_0 is rotated under the mapping $w = z^2$ if
(a) $z_0 = i$, (b) $z_0 = -1/4$, (c) $z_0 = 1 + i$.
Also find the corresponding values of dilatation.

(2) Which part of the plane is shrunk and which part stretched under the following maps: (a) $w = z^2$, (b) $w = z^2 + 2z$, (c) $w = 1/z$?

(3) Find the real and imaginary parts of the following complex numbers:
(a) $\cos(2 + i)$, (b) $\sin 2i$, (c) $\cosh(2 - i)$, (d) $\sinh e^{\pi i}$.

(4) What is the image under the map $w = e^z$ of the straight line $y = k(x - a)$, $k \neq 0, k \neq \infty$?

(5) Prove that $\sin(\frac{\pi}{2} - z) = \cos z$.

(6) Find all periods of $\sin z$. (That is, find all $w \in \mathbb{C}$ such that $\sin(z + w) = \sin z$ for any $z \in \mathbb{C}$.)

(7) Evaluate the integrals $J_1 = \int_L x\,dz$, $J_2 = \int_L y\,dz$ along the following curves:
(a) The line segment joining points $z = 0$ and $z = 2 + i$,
(b) The semicircle $|z| = 1$, $\mathrm{Im} z \geq 0$, with initial point $z = 1$,
(c) The circle $|z - a| = R$.

(8) Evaluate the integral $\int_L |z|\,dz$ along the same paths as in the previous problem.

(9) Evaluate the integral
$$\int_L \frac{z}{\bar{z}} dz,$$
where L is a closed contour that bounds "upper semi-ring" $1 \leq |z| \leq 2$, $\mathrm{Im}\, z \geq 0$, traversed counterclockwise.

(10) Prove that
$$\lim_{r\to 0} \int_{|z-a|=r} \frac{f(z)}{z-a} dz = 2\pi i f(a),$$
if f is continuous in a neighborhood of the point $z = a$. (Hint: in such event, you can write $f(z) = f(a) + \alpha(z)$, where α is continuous in a neighborhood of $z = a$ and $\alpha(z) \to 0$ as $z \to a$, in particular, as $r \to 0$.)